SOME RESULTS ON IDEALS AND $\odot$-DERIVATION IN BL-ALGEBRAS

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Abstract. In this paper, by considering the notion of ideals in BL-algebras, we introduce some special ideals which are related to a subset of a BL-algebra and derive some new relations and results about them. We also define the concepts of $\odot$-derivation for BL-algebras and obtain some related results. Finally, we investigate the connection between these functions and BL-algebras.

1. Introduction

BL-algebras, introduced in 1998 by Hájek, serve as an algebraic framework for basic logic, specifically the logic of continuous t-norms [1]. Due to the fact that fuzzy logic has made significant progress in most branches of science and engineering, most researchers prioritized the study of mathematical fuzzy logic in their research (such as [2, 3, 4]). Therefore, in this context, Hájek introduced BL-algebras based on basic fuzzy logic. Notably, they encompass both the Łindenbaum algebra of equivalent formulas and the algebraic counterpart of propositional basic logic [1]. On the other hand, MV-algebras, initially proposed by C. C. Chang for proving the completeness theorem of Łukasiewicz logic, represents a special class of BL-algebras [5]. While all MV-algebras can be classified as BL-algebras, the converse is not always true. However, Höhle demonstrated that this holds when the double negation law is satisfied, i.e., for every $x \in L$, we have $x = x^{--}$ [6]. Consequently, a BL-algebra can be perceived as an algebraic structure such that in general, the recent equality is not hold in it.

In algebraic structures, filters and ideals are essential and practical tools. In MV-algebras, ideals play a fundamental role, whereas in BL-algebras, filters assume this crucial role.

Despite the vast majority of research on algebraic structures, particularly on BL-algebras, most of studies have primarily focused on filters. As a result, the investigation of ideals in BL-algebras have received comparatively less attention. C. Lele et al. introduced the notion of ideals in BL-algebras and emphasized that it is due to the lack of operation in the rule of double negation and a suitable operation $\oplus$ in BL-algebras, ideals and filters do not exhibit duality [7]. A. Paad expanded on the concepts of integral ideals and radical ideals in BL-algebras, leading to novel findings in this domain [8, 9]. Some authors obtained results from this point of view by using the concept of ideals (such as [10, 7, 11]).

In this paper, our focus lies on the concept of ideals in BL-algebras. We introduce some special ideals associated with subsets of a BL-algebra L and derive new relations and results concerning these ideals. Some authors introduced the concept of φ-derivations in BL-algebras [12, 13]. We define the notion of $\odot$-derivations for BL-algebras, investigating their properties and implications. Furthermore, we delve into the relationship between these functions within the context of BL-algebras.

To order to provide a comprehensive understanding, this paper is structured as follows: In Section 2, we summarize the basic definitions and essential concepts. In Section 3, we introduce the special ideals in BL-algebras and establish theorems and relations about them. Section 4 is dedicated to defining the concept of $\odot$-derivatives in BL-algebras and their properties.

2. Preliminaries

In this section, we will review some key definitions and properties of BL-algebras that will be utilized throughout the paper.

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Definition 2.1. [1] A BL-algebra is an algebra $L = (L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, \odot, \rightarrow$ and two constants $0, 1$ such that:

BL_1: $L = (L, \wedge, \vee, 0, 1)$ is a bounded lattice;

BL_2: $L = (L, \odot, 1)$ is a commutative monoid;

BL_3: $\odot$ and $\rightarrow$ form a adjoint pair, i.e., $r \odot t \leq s \Longleftrightarrow r \leq t \rightarrow s$, for all $r, s, t \in L$;

BL_4: $r \wedge s = r \odot (r \rightarrow s)$, for all $r, s \in L$;

BL_5: $(r \rightarrow s) \vee (s \rightarrow r) = 1$.

If $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra, for every $t \in L$ and a natural number n, we define $t^- = t \rightarrow 0$ and $t^n = t^{n-1} \odot t$, for $n \in \mathbb{N}$, $t^0 = 1$.

The following properties hold in every BL-algebras.

Proposition 2.1. [14, 1, 15] *Let L be a BL-algebra. For all $r, s, t \in L$, and $n \in \mathbb{N}$, the following statements hold:*

(1) $r \odot (r \rightarrow s) \leq s$;
(2) $r \odot s \leq r \wedge s \leq r, s$;
(3) $r \leq s$ *if and only if* $r \rightarrow s = 1$;
(4) $1 \rightarrow r = r,\ r \rightarrow r = 1,\ r \leq s \rightarrow r,\ r \rightarrow 1 = 1,\ 0 \rightarrow r = 1$;
(5) $r \odot r^- = 0$;
(6) $r \leq r^{--},\ 1^- = 0,\ 0^- = 1,\ r^{---} = r^-$;
(7) $r \rightarrow (s \rightarrow t) = s \rightarrow (r \rightarrow t) = (r \odot s) \rightarrow t$;
(8) *If* $r \leq s$, *then* $s \rightarrow t \leq r \rightarrow t$ *and* $t \rightarrow r \leq t \rightarrow s$;
(9) $r \leq s$ *implies* $s^- \leq r^-$;
(10) $(r \odot s)^{--} = r^{--} \odot s^{--}$, $(r \wedge s)^{--} = r^{--} \wedge s^{--}$, $(r \vee s)^{--} = r^{--} \vee s^{--}$, $(r \rightarrow s)^{--} = r^{--} \rightarrow s^{--}$;
(11) $r \odot (s \wedge t) = (r \odot s) \wedge (r \odot t)$, $r \odot (s \vee t) = (r \odot s) \vee (r \odot t)$;
(12) $(r \wedge s)^- = r^- \vee s^-$ *and* $(r \vee s)^- = r^- \wedge s^-$.

Based on references [14, 6], BL-algebras can be characterized as distributive lattices. In a distributive lattice $(L, \leq, \wedge, \vee)$, every element $r \in L$ is associated with an element $r^* \in L$ such that for any $s \in L$, the following conditions hold: $(r \wedge r^*) \vee s = s$ and $(r \vee r^*) \wedge s = s$. Such a distributive lattice with these properties is known as a Boolean algebra. The element r^* is referred to as the lattice complement of r. Moreover, the set of all complemented elements in the corresponding distributive lattice associated with the BL-algebra L forms a Boolean algebra, denoted by $B(L)$.

Theorem 2.1. [14, 1] *For every $r, s \in L$, the following statements are equivalent:*

(i) $r \in B(L)$;
(ii) $r \odot r = r$ *and* $r = r^{--}$;
(iii) $r \odot r = r$ *and* $r^- \rightarrow r = r$;
(iv) $r \vee r^- = 1$;
(v) $(r \rightarrow s) \rightarrow r = r$;
(vi) $r \wedge s = r \odot s$;

Definition 2.2. [7] A subset I of BL-algebra L is called ideal, if it satisfies:

I_1: If $r, s \in I$, then $r \oslash s = r^- \rightarrow s \in I$;

I_2: If $r \in L$, $s \in I$ and $r \leq s$, then $r \in I$.

From [7], $\{0\}$ is the simplest example of ideals, and for every $r \in L$, $r \in I$ if and only if $r^{--} \in I$.

A proper ideal I is called the prime ideal of L, if $(r \rightarrow s)^- \in I$ or $(s \rightarrow r)^- \in I$, for every $r, s \in L$. An ideal I of L is called maximal, if it is proper and no proper ideal of L strictly contains I, i.e., for every ideal $J \neq I$, if $I \subseteq J$, then $J = L$ [7].

Proposition 2.2. [7] *A proper ideal I of a BL-algebra L is a prime ideal if and only if for every $r, s \in L$, $r \wedge s \in I$ implies that $r \in I$ or $s \in I$.*

Theorem 2.2. [7] *A set I containing 0 of a BL-algebra L is an ideal if and only if for every $r, s \in L$, $r^- \odot s \in I$ and $r \in I$ imply $s \in I$.*

Definition 2.3. [1] Let L_1 and L_2 be two BL-algebras. A function $\lambda : L_1 \rightarrow L_2$ is called a BL-homomorphism, if for all $r, s \in L_1$:

H_1: $\lambda(0_{L_1}) = 0_{L_2}$;

H_2: $\lambda(r \odot s) = \lambda(r) \odot \lambda(s)$;

H_3: $\lambda(r \rightarrow s) = \lambda(r) \rightarrow \lambda(s)$.

Remark 2.1. *From [6], if $\lambda : L_1 \rightarrow L_2$ is a BL-homomorphism, then for every $r, s \in L_1$ we have:*

h_1: $\lambda(r \wedge s) = \lambda(r) \wedge \lambda(s)$;

h_2: $\lambda(r \vee s) = \lambda(r) \vee \lambda(s)$;

h_3: $\lambda(r^-) = (\lambda(r))^-$;

h_4: *If* $r \leq s$, *then* $\lambda(r) \leq \lambda(s)$;

h_5: $\lambda(r \oslash s) = \lambda(r) \oslash \lambda(s)$.

If $\lambda : L_1 \rightarrow L_2$ is a BL-homomorphism, then the kernel of λ is defined by, $\ker \lambda = \{r \in L_1 |\ \lambda(r) = 0\}$.

Theorem 2.3. [7] *Let I be an ideal of BL-algebra L. Define the relation $\sim_I$ on L by $r \sim_I s \Leftrightarrow r^- \odot s \in I$ and $s^- \odot r \in I$. Then $\sim_I$ is a congruence on L and $\frac{L}{I} = \{\frac{r}{I} |\ r \in L\}$, where $\frac{r}{I} = \{s \in L |\ r \sim_I s\}$. Moreover, $\frac{L}{I}$ is a BL-algebra.*

For every $\frac{r}{I} \in \frac{L}{I}$, $(\frac{r}{I})^{--} = \frac{r}{I}$. This means that the quotient BL-algebra $\frac{L}{I}$ which is constructed by any ideal is an MV-algebra [10, Proposition 4.3].

Remark 2.2. [7] *For every r, s in BL-algebra L, $(\frac{r}{I})^- = \frac{r^-}{I}$, $(\frac{r}{I}) \oslash (\frac{s}{I}) = \frac{r \oslash s}{I}$ and $\frac{r}{I} = \frac{s}{I} \Leftrightarrow (r^- \odot s), (s^- \odot r) \in I$.*

We observe that, $\frac{r}{I} = \frac{0}{I} \Leftrightarrow r \in I$ and $\frac{r}{I} = \frac{1}{I} \Leftrightarrow r^- \in I$.

Definition 2.4. [1] Let L be a BL-algebras. Then, a function $f : L \to L$ is called isotone, if $r \leq s$ implies that $f(r) \leq f(s)$, for all $r, s \in L$.

3. Special Ideals on BL-algebras

In this section, by considering a subset of a BL-algebra L, we introduce some special ideals and obtain some new relations between them.

Definition 3.1. The set $J(X) = \{a \in L| \ a \to x = 1, \text{ for all } x \in X\}$ is called the adjoint of X, where X is a subset of BL-algebra L.

Example 3.1. *Let $L = \{0, a, b, 1\}$ with $0 < a < b < 1$. Define " $\odot$ " and " $\to$ " as follows:*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

It is easy to see that $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. Let $X = \{0, a, b\}$, $Y = \{b, 1\}$, $Z = \{b\}$ be three subsets of L. So we conclude that $J(X) = \{0\}$, $J(Y) = \{0, a, b\}$ and $J(Z) = \{0, a, b\}$.

Proposition 3.1. *Let X be a subset of a BL-algebra L. Then $J(X)$ is an ideal of L.*

Proof. Since $0 \to r = 1$, for all $r \in X$, $0 \in J(X)$ and $J(X) \neq \phi$. Let $s \in L$, $t \in J(X)$ and $s \leq t$. Then $t \to r = 1$, for all $r \in X$. By Proposition 2.1(8), can write $t \to r \leq s \to r$, for all $r \in X$. This means that $1 \leq s \to r$ and $s \to r = 1$, i.e., $s \in J(X)$. Now suppose $s, t \in J(X)$. We claim that $s \oslash t \in J(X)$. In other words, $s \oslash t \to r = 1$. Suppose it is not, i.e., $s \oslash t \to r \neq 1$, then $s \oslash t > r$. Therefore $r < s^- \to t$. Then $r \odot s^- < t$. By Proposition 2.1(8), we conclude $t \to r < r \odot s^- \to r$. But by assumption $t \in J(X)$, then $t \to r = 1$. This means that $1 < r \odot s^- \to r$ and this is a contradiction. Therefore $J(X)$ is an ideal. □

Theorem 3.1. *Let L be a BL-algebra and $X, Y \subseteq L$. Then:*

(i) If $X \subseteq Y$, then $J(Y) \subseteq J(X)$;
(ii) $J(X \cup Y) = J(X) \cap J(Y)$;
(iii) $t \in J(X)$ if and only if $t^n \to x = 1$, for every $x \in X$ and $n \in \mathbb{N}$.

Proof. *(i)* Let $a \in J(Y)$. Then $a \to y = 1$, for every $y \in Y$. We suppose that $t \in X$. Since $X \subseteq Y$, $t \in Y$, i.e., $a \to t = 1$, thus, $a \in J(X)$.
(ii) By *(i)*, $J(X \cup Y) \subseteq J(X)$ and $J(X \cup Y) \subseteq J(Y)$. Therefore $J(X \cup Y) \subseteq J(X) \cap J(Y)$. Now, if t is an element of the second side, then $t \to x = 1$ and $t \to y = 1$ for every $x \in X$ and $y \in Y$. We get $t \to c = 1$ for every $c \in X \cup Y$.
(iii) Let $t \in J(X)$. Then $t \to x = 1$, for every $x \in X$. From Proposition 2.1(10), $t^n \to x = (t^{n-1} \odot t) \to x = t^{n-1} \to (t \to x)$. Therefore $t^n \to x = t^{n-1} \to 1 = 1$. Conversely, if $t^n \to x = 1$, for every $n \in \mathbb{N}$, then, $t \to x = 1$. □

We define the set $J(X_a)$ by $J(X_a) = \{b \in L| \ (b \to a)^- \in J(X)\}$, for every subset X and element a in BL-algebra L. It is clear that $0, a \in J(X_a)$. In Example 3.1, $J(X_a) = \{0, a\}$, $J(Y_a) = \{0, a, b, 1\}$.

Proposition 3.2. *Let L be a BL-algebra and $X \subseteq L$. Then $J(X_a)$ is an ideal of L.*

Proof. $0 \in J(X_a)$. Let $c \in J(X_a)$ and $b \leq c$. Thus $(c \to a)^- \in J(X)$. By Proposition 2.1(9), $c \to a \leq b \to a$ and $(b \to a)^- \leq (c \to a)^-$. Since $(c \to a)^- \in J(X)$ and $J(X)$ is an ideal, $(b \to a)^- \in J(X)$, i.e., $b \in J(X_a)$.
Now suppose $b, c \in J(X_a)$. Similar to the proof of Proposition 3.1, it can be shown that $b \oslash c \in J(X_a)$. Then $J(X_a)$ is an ideal. □

Theorem 3.2. *If L is a BL-algebra and a, b are elements of L, then the following statements hold for every subsets X, Y of L:*

(i) $J(X) \subseteq J(X_t)$, for every $t \in X$;
(ii) If $a \leq b$, then $J(X_a) \subseteq J(X_b)$;
(iii) If $J(X) \subseteq J(Y)$, then $J(X_a) \subseteq J(Y_a)$;
(iv) $J(X_a) \cap J(Y_a) = J((X \cup Y)_a)$, $J(X_a) \cup J(Y_a) = J((X \cap Y)_a)$.

Proof. *(i)* Let $b \in J(X)$. Then by Definition 3.1, $b \to x = 1$ for every $x \in X$. Therefore $b \to t = 1$. We conclude that, $(b \to t)^- = 1^- = 0 \in J(X)$, i.e., $b \in J(X_t)$.
(ii) Let $r \in J(X_a)$. Then, $(r \to a)^- \in J(X)$. Assuming $a \leq b$. Then by Proposition 2.1(8), $r \to a \leq r \to b$ for every $r \in X$. Therefore $(r \to b)^- \leq (r \to a)^-$ for every $r \in X$. Since $J(X)$ is an ideal, $(r \to b)^- \in J(X)$. This means that $r \in J(X)_b$.
(iii) Let $t \in J(X_a)$. Then $(t \to a)^- \in J(X) \subseteq J(Y)$. Thus $(t \to a)^- \in J(Y)$, i.e., $t \in J(Y_a)$.
(iv) We know $J(X \cup Y) = J(X) \cap J(Y) \subseteq J(X), J(Y)$. By *(iii)*, $J((X \cup Y)_a) \subseteq J(X_a), J(Y_a)$. Therefore $J((X \cup Y)_a) \subseteq J(X_a) \cap J(Y_a)$. Conversely, let $t \in J(X_a) \cap J(Y_a)$. Then $t \in J(X_a)$, $t \in J(Y_a)$. This means that, $(t \to a)^- \in J(X)$, $(t \to a)^- \in J(Y)$, hence $(t \to a)^- \in J(X) \cap J(Y)$, i.e., $t \in J((X \cup Y)_a)$. In the similar way, it is clear $J(X_a) \cup J(Y_a) = J((X \cap Y)_a)$. □

Definition 3.2. *Let L be a BL-algebra and $X \subseteq L$. We define the set $D(X)$ by $D(X) = \{a \in L \mid a^{--} \to x = 1$ for all $x \in X\}$ and we call double adjoint X.*

Example 3.2. *We consider the BL-algebra L in [6, Example 3.5]. It is clear that, $0^{--} = 0, a^{--} = b^{--} = c^{--} = c$, $d^{--} = d$ and $e^{--} = f^{--} = 1$. Let $X = \{f, e\}$. Then $D(X) = \{0, d\}$ and $D(a) = \{0\}$.*

Proposition 3.3. *$D(X)$ is an ideal, for every subset X of BL-algebra L.*

Proof. It is easy to see that $0 \in D(X)$. We assume $r \in L$, $s \in D(X)$ and $r \leq s$. Then $s^{--} \to x = 1$, for every $x \in X$. By Proposition 2.1(9), $s^- \leq r^-$. Therefore $r^{--} \leq s^{--}$ and $s^{--} \to x \leq r^{--} \to x$. We conclude $1 \leq r^{--} \to x$, i.e., $r^{--} \to x = 1$ and $r \in D(X)$.
Suppose $r, s \in D(X)$. We have to prove $r \oslash s \in D(X)$. Suppose it is not, i.e., $(r \oslash s)^{--} \to x \neq 1$, for some $x \in X$, then $x < r^{---} \to s^{--}$. By Proposition 2.1(8), $x < r^- \to s^{--}$ and $r^{--} \to x < r^{--} \to (r^- \to s^-)$. Since $r \in D(X)$, $r^{--} \to x = 1$. Therefore $1 < r^{--} \to (r^- \to s^-)$ and this is a contradiction. Then $r \oslash s \in D(X)$, i.e., $D(X)$ is an ideal. □

Theorem 3.3. *Let L be a BL-algebra and $X, Y \subseteq L$. Then the following statements hold:*
(i) $J(1) = D(1) = L$;
(ii) $D(X) \subseteq J(X)$;
(iii) If $a^- \to a = a$, for all $a \in L$, then $J(X) \subseteq D(X)$;
(iv) If $X \subseteq Y$, then $D(Y) \subseteq D(X)$.

Proof. *(i)* We know that, $J(1) = \{a \in L \mid a \to 1 = 1\} = L$ and $D(1) = \{a \in L \mid a^{--} \to 1 = 1\} = L$.
(ii) Let $r \in D(X)$. Then by Definition 3.2, $r^{--} \to x = 1$, for all $x \in X$. Since $r \leq r^{--}$, by Proposition 2.1(8), $r^{--} \to x \leq r \to x$ and $1 \leq r \to x$, i.e., $r \to x = 1$. It follows that $r \in J(X)$.
(iii) Let $r \in J(X)$. Then $r \to x = 1$, for all $x \in X$. Since $0 \leq r$, by Proposition 2.1(8), $r^- \to 0 \leq r^- \to r$, i.e., $r^- \to 0 \leq r$. Therefore $r^{--} \leq r$ and $r \to x \leq r^{--} \to x$, i.e., $1 \leq r^{--} \to x$. Hence $r^{--} \to x = 1$ and $r \in D(X)$.
(iv) Let $r \in D(Y)$. Then $r^{--} \to y = 1$, for all $y \in Y$. Since $X \subseteq Y$, for every $x \in X$, $r^{--} \to x = 1$, i.e., $r \in D(X)$. □

Theorem 3.4. *Let X be a subset of a BL-algebra L. Then:*
(i) $D(X) = \bigcap_{x \in X} D(x)$;
(ii) If $r \in D(x)$, then $x \wedge r^{--} = r^{--}$, for every $x \in X$;
(iii) If $f : L \to L$ is a BL-homomorphism, then $f(J(x)) \subseteq J(f(x))$, for all $x \in L$.

Proof. *(i)* We have:

$$\begin{aligned} t \in D(X) &\Leftrightarrow \quad t^- \to x = 1, \quad for\ every\ x \in X \\ &\Leftrightarrow \quad t \in D\{x\}, \quad for\ every\ x \in X \\ &\Leftrightarrow \quad t \in \bigcap_{x \in X} D\{x\}. \end{aligned}$$

(ii) If $r \in D\{x\}$, then $r^{--} \to x = 1$. From BL_4, we conclude that, $x \wedge r^{--} = r^{--} \wedge x = r^{--} \odot (r^{--} \to x) = r^{--} \odot 1 = r^{--}$.
(iii) Let $y \in f(J(X))$. Then $y = f(a)$, for some $a \in J(X)$. Since $a \to x = 1$, $f(a \to x) = f(a) \to f(x) = f(1) = 1$, i.e., $y \to f(x) = 1$. Therefore $y \in J(f(x))$. □

Theorem 3.5. *Let L_1 be a BL-algebra and $X, Y \subseteq L$. Then $D(X \cup Y) = D(X) \cap D(Y) \subseteq D(X \cap Y)$.*

Proof. We know that, $r \in D(X \cup Y)$

$$\begin{aligned} &\Leftrightarrow \quad r^{--} \to z = 1, for\ every\ z \in X \cup Y \\ &\Leftrightarrow \quad r^{--} \to x = r^{--} \to y = 1, for\ every\ x \in X\ and\ y \in Y \\ &\Leftrightarrow \quad r \in D(X)\ and\ r \in D(Y) \\ &\Leftrightarrow \quad r \in D(X) \cap D(Y). \end{aligned}$$

Since $X \cap Y \subseteq X \cup Y$, by Theorem 3.3*(iv)*, $D(X \cup Y) \subseteq D(X \cap Y)$. $\square$

Theorem 3.6. *Let L be a BL-algebra and $X \subseteq L$. If I is an ideal of L, then $D(\frac{x}{I}) = \{\frac{r}{I} |\ (r^{--} \to x)^{-} \in I,\ \forall x \in X\}$.*

Proof. Since $D(\frac{x}{I}) = \{\frac{r}{I} |\ (\frac{r}{I})^{--} \to (\frac{x}{I}) = \frac{1}{I},\ for\ all\ x \in X\}$, $\frac{r^{--}}{I} \to \frac{x}{I} = \frac{r^{--} \to x}{I} = \frac{1}{I}$. By Remark 2.2, we conclude that $(r^{--} \to x)^{-} \in I$.
We set $D(I) = \{r \in L |\ r^{--} \in I\}$, where I is an ideal of BL-algebra L. If we consider the ideal $I = \{0, a, b, c\}$ of BL-algebra L in [6, Example 3.5], then we get $D(I) = \{0, a, b, c\}$. $\square$

Theorem 3.7. *Let I and J be two ideals of BL-algebra L. Then the following statements hold:*

(i) $D(I) \subseteq I$;
(ii) If $I \subseteq J$, then $D(I) \subseteq D(J)$;
(iii) $D(I \cap J) = D(I) \cap D(J)$;
(iv) $\frac{L}{D(I \cap J)} = \frac{L}{D(I)} \cap \frac{L}{D(J)}$.

Proof. *(i)* Let $r \in D(I)$. Then $r^{--} \in I$. By Definition 2.2 and the fact that $r \leq r^{--}$, we have $r \in I$.
(ii) Let $r \in D(I)$. Then $r^{--} \in I$. Since $I \subseteq J$, $r^{--} \in J$, i.e., $r \in D(J)$.
(iii) We have,

$$\begin{aligned} r \in D(I \cap J) \quad &\Leftrightarrow \quad r^{--} \in I \cap J \\ &\Leftrightarrow \quad r^{--} \in I \ and \ r^{--} \in J \\ &\Leftrightarrow \quad r \in D(I) \ and \ r \in D(J) \\ &\Leftrightarrow \quad r \in D(I) \cap D(J). \end{aligned}$$

(iv) By applying (iv), to quotient BL-algebra, we have $\dfrac{L}{D(I \cap J)} = \dfrac{L}{D(I) \cap D(J)}$.
Therefore,

$$[r] = \frac{r}{D(I) \cap D(J)} \in \frac{L}{D(I) \cap D(J)}$$
$$\Leftrightarrow [r] = \{s \in L |\ r \equiv_{D(I) \cap D(J)} s\}$$
$$\Leftrightarrow [r] = \{s \in L |\ (r^{-} \odot s) \vee (s^{-} \odot r) \in D(I) \cap D(J)\}$$
$$\Leftrightarrow [r] = \{s \in L |\ (r^{-} \odot s) \vee (s^{-} \odot r) \in D(I)\} \cap \{s \in L | (r^{-} \odot s) \vee (s^{-} \odot r) \in D(J)\}$$
$$\Leftrightarrow [r] \in \frac{L}{D(I)} \cap \frac{L}{D(J)}.$$ $\square$

We recall that, if L_1 and L_2 are two BL-algebras then, $L_1 \times L_2 = \{(a, b) |\ a \in L_1, b \in L_2\}$ is a BL-algebra [16].

Theorem 3.8. *If I and J are ideals of BL-algebras L_1 and L_2 respectively, then the following statements hold:*

(i) $I \times J$ is an ideal of $L_1 \times L_2$;
(ii) $D(I \times J) = D(I) \times D(J)$;
(iii) $\frac{L_1 \times L_2}{D(I \times J)} \simeq \frac{L_1}{D(I)} \times \frac{L_2}{D(J)}$;
(iv) If $J \subseteq D(I)$, then $D(\frac{I}{J}) = \frac{D(I)}{J}$.

Proof. *(i)* It is trivial that $(0, 0) \in I \times J$. Now we suppose that $(r, s), (t, z) \in I \times J$. Then $r, t \in I$ and $s, z \in J$. Since I and J are ideals, $r \oslash t \in I$, $s \oslash z \in J$, i.e., $(r \oslash t, s \oslash z) \in I \times J$.
Let $(r, s) \leq (t, z)$ and $(t, z) \in I \times J$. Then $r \leq t$, $s \leq z$, $t \in I$ and $z \in J$. This means that, $r \in I$ and $s \in J$, hence $(r, s) \in I \times J$.
(ii) We know that $D(I \times J) = \{(r, s) |\ (r, s)^{--} \in I \times J;\ r \in L_1,\ s \in L_2\}$. Therefore,

$$\begin{aligned} (r, s)^{--} \in I \times J \quad &\Leftrightarrow \quad (r^{--}, s^{--}) \in I \times J \\ &\Leftrightarrow \quad r^{--} \in I \ and \ s^{--} \in J \\ &\Leftrightarrow \quad r \in D(I) \ and \ s \in D(J) \\ &\Leftrightarrow \quad (r, s) \in D(I) \times D(J). \end{aligned}$$

(iii) We define a mapping $\varphi : L_1 \times L_2 \to \frac{L_1}{D(I)} \times \frac{L_2}{D(J)}$ by $\varphi(r, s) = ([r], [s])$. It is easy to see that φ is an onto BL-homomorphism. Moreover,

$$\begin{aligned} \ker \varphi &= \{(r, s) |\ \varphi(r, s) = ([0], [0])\} \\ &= \{(r, s) |\ ([r], [s]) = ([0], [0])\} \\ &= \{(r, s) |\ [r] = [0], [s] = [0]\} \\ &= \{(r, s) |\ \frac{r}{D(I)} = [0],\ \frac{s}{D(J)} = [0]\}. \end{aligned}$$

By Remark 2.2, $\ker \varphi = \{(r, s) |\ r \in D(I), s \in D(J)\} = \{(r, s) |\ (r, s) \in D(I) \times D(J)\}$. By (ii), $\ker \varphi = D(I) \times D(J) = D(I \times J)$.
(iv) Since $J \subseteq D(I) \subseteq I$, $D(\frac{I}{J}) = \{[r] \in \frac{L_1}{J} |\ [r]^{--} \in \frac{I}{J}\} = \{[r] \in \frac{L_1}{J} |\ [r^{--}] \in \frac{I}{J}\}$. By Definition 2.2, $D(\frac{I}{J}) = \{[r] \in \frac{L_1}{J} |\ r^{--} \in I\} = \{[r] \in \frac{L_1}{J} |\ r \in D(I)\} = \dfrac{D(I)}{J}$. $\square$

Theorem 3.9. *Let I be an ideal of BL-algebra of L. Then:*

(i) If I is a prime ideal of L, then $D(I)$ is a prime ideal of L.
(ii) If $D(I)$ is a maximal ideal, then I is a maximal ideal of L.

Proof. *(i)* Let I be a prime ideal of L and $r \wedge s \in D(I)$. Then $(r \wedge s)^{--} = r^{--} \wedge s^{--} \in I$. Since I is a prime ideal, $r^{--} \in I$ or $s^{--} \in I$. This

means that $r \in D(I)$ or $s \in D(I)$. Hence $D(I)$ is a prime ideal of L.
(ii) Let $D(I)$ be a maximal ideal. Since $D(I) \subseteq I$, $D(I) = I$. Thus I is a maximal ideal of L. $\square$

Theorem 3.10. *Let L_1 and L_2 be two BL-algebras, I an ideal of L_1 and $f : L_1 \to L_2$ be a BL-homomorphism. Then $f(D(I)) \subseteq D(f(I))$.*

Proof. Let $y \in f(D(I))$. Then $y = f(r)$ for some $r \in D(I)$. This means that $r^{--} \in I$ and $y^{--} = (f(r))^{--} = f(r^{--}) \in f(I)$. Therefore $y \in D(f(I))$. $\square$

Corollary 3.1. *Let L_1 and L_2 be two BL-algebra, J an ideal of L_2 and $f : L_1 \to L_2$ be a BL-homomorphism. Then $D(f^{-1}(J)) = f^{-1}(D(J))$.*

Proof. We know that,

$$\begin{aligned} t \in D(f^{-1}(J)) &\Leftrightarrow t^{--} \in f^{-1}(J) \\ &\Leftrightarrow f(t^{--}) \in J \\ &\Leftrightarrow (f(t))^{--} \in J \\ &\Leftrightarrow f(t) \in D(J) \\ &\Leftrightarrow t \in f^{-1}(D(J)). \end{aligned}$$
$\square$

Corollary 3.2. *Let I and J be two ideals of BL-algebra L and $J \subseteq I$. Then:*
(i) $D(I/D(J)) = D(I)/D(J) \subseteq D(I/J)$;
(ii) $D(D(I)) \subseteq D(I)$.

Proof. (i) Since $J \subseteq I$, by Theorem 3.7, $D(J) \subseteq D(I) \subseteq I$. Since $D(J) \subseteq J$, by Theorem 3.8, $D(I)/D(J) \subseteq D(I)/J = D(I/J)$.
(ii) By Theorem 3.7, it is trivial. $\square$

4. $\odot$-DERIVATIONS ON BL-ALGEBRAS

In this section, we introduced some special ideals and the notion of $\odot$-derivations on BL-algebras and we obtain some new results.

Definition 4.1. Let L be a BL-algebra and $\varphi : L \to L$ be a non-identity function. φ is a $\odot$-derivation on L if for every $a, b \in L$:
$\varphi(a \odot b) = (\varphi(a) \odot b^{--}) \vee (\varphi(a^-) \odot b)$.

Example 4.1. *Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define "$\odot$", "$\to$" as follow:*

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra, we define $\varphi_1, \varphi_2 : L \to L$:
(i) $\varphi_1(0) = \varphi_1(1) = 0,\ \varphi_1(a) = a,\ \varphi_1(b) = b$;
(ii) $\varphi_2(0) = \varphi_2(1) = 0,\ \varphi_2(a) = b,\ \varphi_2(b) = a$.
It is clear that φ_1 and φ_2 are $\odot$-derivations on L.

Theorem 4.1. *Let φ be a $\odot$-derivation on BL-algebra L. Then the following statements hold, for all $a, b \in L$:*
(i) $\varphi(0) = 0$;
(ii) $a \le b$ implies $\varphi(a) \le b^{--}$, $\varphi(a^-) \le b^{--}$;
(iii) $\varphi(a) \le a^{--}$, $\varphi(a^-) \le a^{--}$;
(iv) $\varphi(1) = 0$ and if $\varphi(a) = 1$, then $a^- = 0$;
(v) $\varphi(a^-) \le \varphi(a)$;
(vi) $\varphi(a^-) \le (\varphi(a))^-$.

Proof. *(i)* We know $\varphi(0) = \varphi(0 \odot 0)$. By Definition 4.1, we have, $\varphi(0 \odot 0) = (\varphi(0) \odot 0) \vee (\varphi(1) \odot 0) = 0 \vee 0 = 0$. Therefore $\varphi(0) = 0$.
(ii) Since $a \le b$, $a \odot b^- = 0$. By Definition 4.1, $0 = \varphi(0) = \varphi(a \odot b^-) = (\varphi(a) \odot b^{---}) \vee (\varphi(a^-) \odot b^-)$. Then $(\varphi(a) \odot b^-) \vee (\varphi(a^-) \odot b^-) = 0$, i.e, $\varphi(a) \odot b^- = 0$ and $\varphi(a^-) \odot b^- = 0$. This means that $\varphi(a) \le b^{--}$ and $\varphi(a^-) \le b^{--}$.
(iii) By (ii) and taking $a = b$, it is clear.
(iv) By (ii), by taking $a = 0$, $\varphi(0^-) \le 0^{--}$, hence $\varphi(1) \le 0$, i.e, $\varphi(1) = 0$. By (iii) and assumption $\varphi(a) = 1$, therefore $1 \le a^{--}$, i.e., $a^{--} = 1$. This means that $a^- = 0$.
(v) By Definition 4.1, $\varphi(a) = \varphi(a \odot 1) = (\varphi(a) \odot 1) \vee (\varphi(a^-) \odot 1) = \varphi(a) \vee \varphi(a^-)$. Then $\varphi(a^-) \le \varphi(a)$.
(vi) By (iii), $\varphi(a) \le a^{--}$, hence $\varphi(a^-) \le a^{---}$ and $a^{---} \le (\varphi(a))^-$. Therefore $\varphi(a^-) \le (\varphi(a))^-$. $\square$

Theorem 4.2. *Let φ be a $\odot$-derivation on BL-algebra L. Then the following statements hold for all $a, b \in B(L)$:*
(i) $\varphi(a) \odot \varphi(b) \le \varphi(a \odot b)$;
(ii) $\varphi(a \wedge b) = b \wedge (\varphi(a) \vee \varphi(a^-))$.

Proof. *(i)* By Theorem 4.1(iii), $\varphi(b) \le b^{--}$, hence $\varphi(a) \odot \varphi(b) \le \varphi(a) \odot b^{--} \le (\varphi(a) \odot b) \vee (\varphi(a) \odot b^{--}) = \varphi(a \odot b)$.
(ii) By Theorem 2.1, $a \wedge b = a \odot b$ *and* $a^{--} = a$. Then:

$$\begin{aligned} \varphi(a \wedge b) = \varphi(a \odot b) &= (\varphi(a^-) \odot b) \vee (\varphi(a) \odot b^{--}) \\ &= (\varphi(a^-) \wedge b) \vee (\varphi(a) \wedge b) \\ &= b \wedge (\varphi(a) \vee \varphi(a^-)). \end{aligned}$$
$\square$

Theorem 4.3. *Let φ be a $\odot$-derivation on BL-algebra L. If $a \odot b = 0$, then $\varphi(a) \odot \varphi(b) = 0$.*

Proof. By Theorem 4.1(*iii*), $\varphi(a) \leq a^{--}$. Then $\varphi(a) \odot \varphi(b) \leq a^{--} \odot \varphi(b) \leq a^{--} \odot b^{--} = (a \odot b)^{--} = 0^{--} = 0$, i.e., $\varphi(a) \odot \varphi(b) = 0$. $\square$

Theorem 4.4. *Let φ be a $\odot$-derivation on BL-algebra L. Then:*

$$\varphi(a^- \wedge b^-) \leq \varphi(a \vee b)\ , \quad \varphi(a^- \vee b^-) \leq \varphi(a \wedge b).$$

Proof. We know that $\varphi(a^-) \leq \varphi(a)$, then $\varphi(a \wedge b)^- \leq \varphi(a \wedge b)$ *and* $\varphi(a \vee b)^- \leq \varphi(a \vee b)$. By Proposition 2.1(14), we have $\varphi(a^- \vee b^-) \leq \varphi(a \wedge b)$ *and* $\varphi(a^- \wedge b^-) \leq \varphi(a \vee b)$. $\square$

Theorem 4.5. *φ is a $\odot$-derivation on BL-algebra L if and only if $\varphi(a \odot b) = \varphi(a) \odot b^{--}$ for all $a, b \in L$.*

Proof. Let φ be a $\odot$-derivation. By part (v) of Theorem 4.1, since $\varphi(a^-) \leq \varphi(a)$, $\varphi(a^-) \odot b \leq \varphi(a) \odot b$ and since $b \leq b^{--}$, we have $\varphi(a^-) \odot b \leq \varphi(a) \odot b^{--}$. Therefore $(\varphi(a^-) \odot b) \vee (\varphi(a) \odot b^{--}) = \varphi(a) \odot b^{--}$, i.e., $\varphi(a \odot b) = \varphi(a) \odot b^{--}$.
Conversely, now let $\varphi(a \odot b) = \varphi(a) \odot b^{--}$, for all $a, b \in L$. By Theorem 4.1, since $\varphi(a^-) \leq \varphi(a)$, $\varphi(a^-) \odot b \leq \varphi(a) \odot b \leq \varphi(a) \odot b^{--}$. Therefore $(\varphi(a^-) \odot b) \vee (\varphi(a) \odot b^{--}) = \varphi(a) \odot b^{--} = \varphi(a \odot b)$. Thus φ is a $\odot$-derivation. $\square$

Theorem 4.6. *Let φ be a $\odot$-derivation on BL-algebra L. Then the following statements hold, for all $a, b, c \in L$, $n \in \mathbb{N}$:*
(i) $\varphi(a^n) = \varphi(a) \odot (a^{--})^{n-1}$;
(ii) $\varphi^n(a \odot b) = \varphi^n(a) \odot b^{--}$;
(iii) $\varphi(a \odot b) \odot \varphi(c \odot b^-) = 0$.

Proof. *(i)* The proof is by induction on n. For $n = 1$, we have $\varphi(a) = \varphi(a) \odot 1 = \varphi(a) \odot (a^{--})^0$. Now let for $n = k$, $\varphi(a^k) = \varphi(a) \odot (a^{--})^{k-1}$. Putting $n = k + 1$. Then $\varphi(a^{k+1}) = \varphi(a^k \odot a) = \varphi(a^k) \odot a^{--} = (\varphi(a) \odot (a^{--})^{k-1}) \odot a^{--} = \varphi(a) \odot ((a^{--})^{k-1} \odot a^{--}) = \varphi(a) \odot (a^{--})^k$.
(ii) The proof is by induction on n. For $n = 1$, by Theorem 4.7, it is clear. Now, assume $\varphi^k(a \odot b) = \varphi^k(a) \odot b^{--}$ for each positive integer k. Let $n = k + 1$. Then $\varphi^{k+1}(a \odot b) = \varphi(\varphi^k(a \odot b)) = \varphi(\varphi^k(a) \odot b^{--}) = \varphi^{k+1}(a) \odot b^{----} = \varphi^{k+1}(a) \odot b^{--}$.
(iii) By Theorem 4.7, $\varphi(a \odot b) \odot \varphi(c \odot b^-) = (\varphi(a) \odot b^{--}) \odot (\varphi(c) \odot b^{---}) = (\varphi(a) \odot b^{--}) \odot (\varphi(c) \odot b^-) = (\varphi(a) \odot \varphi(c)) \odot (b^{--} \odot b^-) = (\varphi(a) \odot \varphi(c)) \odot 0 = 0$. $\square$

Theorem 4.7. *Let φ_1 and φ_2 be two $\odot$-derivations on BL-algebra. Then $\varphi_1 \circ \varphi_2$ is a $\odot$-derivation.*

Proof. Let φ_1 and φ_2 be two $\odot$-derivations. By Theorem 4.7, $(\varphi_1 \circ \varphi_2)(a \odot b) = \varphi_1(\varphi_2(a \odot b)) = \varphi_1(\varphi_2(a) \odot b^{--}) = \varphi_1(\varphi_2(a)) \odot b^{----}$, i.e., $(\varphi_1 \circ \varphi_2)(a \odot b) = (\varphi_1 \circ \varphi_2)(a) \odot b^{--}$. Therefore $\varphi_1 \circ \varphi_2$ is a $\odot$-derivation. $\square$

Definition 4.2. Let L be a BL-algebra and $a \in L$. We define the function $\varphi_a : L \to L$ by $\varphi_a(r) = a \odot r^{--}$, for all $r \in L$.

Theorem 4.8. *Let L be a BL-algebra. Then the following statements hold, for all $r, s \in L$:*
(i) φ_a *is a $\odot$-derivation;*
(ii) $\varphi_0(r) = 0$, $\varphi_1(r) = r^{--}$;
(iii) $\varphi_{a\wedge b}(r) = \varphi_a(r) \wedge \varphi_b(r)$, $\varphi_{a\vee b}(r) = \varphi_a(r) \vee \varphi_b(r)$;
(iv) $\varphi_a(r \wedge s) = \varphi_a(r) \wedge \varphi_a(s)$, $\varphi_a(r \vee s) = \varphi_a(r) \vee \varphi_a(s)$;
(v) $\varphi_a(r^n) = \varphi_a(r) \odot (r^{--})^{n-1}$;
(vi) $(\varphi_a \circ \varphi_b)(r) = \varphi_{(\varphi_a(b))}(r)$.

Proof. *(i)* By Definition 4.2, $\varphi_a(r \odot s) = a \odot (r \odot s)^{--}$. By Proposition 2.1(11), $\varphi_a(r \odot s) = a \odot (r^{--} \odot s^{--}) = (a \odot r^{--}) \odot s^{--} = \varphi_a(r) \odot s^{--}$.
(ii) By Definition 4.2, $\varphi_0(r) = 0 \odot r^{--} = 0$ and $\varphi_1(r) = 1 \odot r^{--} = r^{--}$.
(iii) By Definition 4.2, $\varphi_{a\wedge b}(r) = (a \wedge b) \odot r^{--} = r^{--} \odot (a \wedge b)$. By Proposition 2.1(13), $\varphi_{a\wedge b}(r) = (r^{--} \odot a) \wedge (r^{--} \odot b) = (a \odot r^{--}) \wedge (b \odot r^{--}) = \varphi_a(r) \odot \varphi_b(r)$. In the similar way, we can prove, $\varphi_{a\vee b}(r) = \varphi_a(r) \vee \varphi_b(r)$.
(iv) By hypothesis, $\varphi_a(r \wedge s) = a \odot (r \wedge s)^{--}$. By Proposition 2.1(11), (13), $\varphi_a(r \wedge s) = a \odot (r^{--} \wedge s^{--}) = (a \odot r^{--}) \wedge (a \odot s^{--}) = \varphi_a(r) \wedge \varphi_a(s)$. It is proved similarly way, $\varphi_a(r \vee s) = \varphi_a(r) \vee d_a(s)$.
(v) The proof is by induction on n. For $n = 1$, we have $\varphi_a(r) = \varphi_a(r) \odot 1 = \varphi_a(r) \odot (r^{--})^0$. Now suppose $\varphi_a(r^k) = \varphi_a(r) \odot (r^{--})^{k-1}$, for each positive integer k. Let $n = k + 1$. Then $\varphi_a(r^{k+1}) = \varphi_a(r^k \odot r) = \varphi_a(r^k) \odot r^{--}$. Assuming, $\varphi_a(r^{k+1}) = (\varphi_a(r) \odot (r^{--})^{k-1}) \odot r^{--} = \varphi_a(r) \odot ((r^{--})^{k-1} \odot r^{--}) = \varphi_a(r) \odot (r^{--})^k$.

(vi) We have $(\varphi_a \circ \varphi_b)(r) = \varphi_a(\varphi_b(r))$. By Definition 4.2,

$$\begin{aligned}(\varphi_a \circ \varphi_b)(r) &= a \odot (\varphi_b(r))^{--} \\ &= a \odot (b \odot r^{--})^{--} \\ &= a \odot (b^{--} \odot r^{----}) \\ &= a \odot (b^{--} \odot r^{--}) \\ &= (a \odot b^{--}) \odot r^{--} \\ &= \varphi_a(b) \odot r^{--} \\ &= \varphi_{(\varphi_a(b))}(r). \qquad \square\end{aligned}$$

Definition 4.3. Let L be a BL-algebra and assume that $\lambda : L \to L$ be a BL- homomorphism. A function $\varphi : L \to L$ is called a λ-derivation, if for all $a, b \in L$, $\varphi(a \odot b) = \lambda(b) \odot (\varphi(a) \vee \varphi(b))$

Example 4.2. *If we consider the BL-algebra in Example 3.1, $\lambda = I$ and $\varphi : L \to L$ by $\varphi(b) = a$, $\varphi(a) = a, \varphi(0) = 0$ and $\varphi(1) = 1$, then φ is a λ-derivation, since $\varphi(a \odot b) = \varphi(a) = a$ and $\lambda(b) \odot (\varphi(a) \vee \varphi(b)) = b \odot (a \vee a) = b \odot a = a \odot b = a$.*

Theorem 4.9. *Let φ be a λ-derivation on BL-algebra L. Then for every $a, b \in L$:*

(i) $\varphi(0) = 0$;

(ii) $\varphi(a^n) = \lambda(a^n) \odot \varphi(a)$, *for* $n \geq 2$;

(iii) If $a \leq b$, *then* $\varphi(a) \leq \lambda^{--}(b)$. *In particular,* $\varphi(a) \leq \lambda^{--}(a)$;

(iv) $\varphi(1) \leq \varphi(a)$;

(v) If $\varphi(1) = 1$, *then* $\varphi(a) = 1$;

(vi) $a \leq b$ *implies* $\varphi(a) \odot \lambda(b) = \lambda(a)$.

Proof. By Definition 4.3, we have:
(i) $\varphi(0) = \varphi(0 \odot 0) = \lambda(0) \odot (\varphi(0) \vee \varphi(0)) = 0 \odot \varphi(0) = 0$.
(ii) The proof is by induction on n. For $n = 2$, we have, $\varphi(a^2) = \varphi(a \odot a) = \lambda(a) \odot (\varphi(a) \vee \varphi(a)) = \lambda(a) \odot \varphi(a)$. Now suppose $\varphi(a^k) = \lambda(a^k) \odot \varphi(a)$. Therefore $\varphi(a^k) \leq \varphi(a)$. Let $n = k + 1$, thus $\varphi(a^{k+1}) = \varphi(a \odot a^k) = \lambda(a^k) \odot (\varphi(a) \vee \varphi(a^k))$. Since $\varphi(a^k) \leq \varphi(a)$, $\varphi(a^k) \vee \varphi(a) = \varphi(a)$. Therefore $\varphi(a^{k+1}) = \lambda(a^k) \odot \varphi(a)$.
(iii) If $a \leq b$, then $a \odot b^- = 0$. By (i), $0 = \varphi(0) = \varphi(a \odot b^-) = \lambda(b^-) \odot (\varphi(a) \vee \varphi(b^-))$. By Proposition 2.1(13), $0 = (\lambda(b^-) \odot \varphi(a)) \vee (\lambda(b^-) \odot \varphi(b^-))$. Therefore $\lambda(b^-) \odot \varphi(a) = 0$ *and* $\lambda(b^-) \odot \varphi(b^-) = 0$. Since $\lambda(b^-) = (\lambda(b))^- = \lambda^-(b)$, $\lambda^-(b) \odot \varphi(a) = 0$ and $\lambda^-(b) \odot \varphi(b^-) = 0$. Thus $\varphi(a) \leq \lambda^{--}(b)$. Now by taking $a = b$, we have $\varphi(a) \leq \lambda^{--}(a)$.
(iv) By Definition 4.3, $\varphi(a) = \varphi(a \odot 1) = \lambda(1) \odot (\varphi(a) \vee \varphi(1)) = 1 \odot (\varphi(a) \vee \varphi(1)) = \varphi(a) \vee \varphi(1)$, i.e., $\varphi(1) \leq \varphi(a)$.
(v) By (iv), $\varphi(1) \leq \varphi(a)$. Then $1 \leq \varphi(a)$. Therefore $\varphi(a) = 1$.
(vi) Since $a \leq b$, $a \wedge b = a$. Then $\varphi(a) = \varphi(a \wedge b) = \varphi(b \odot (b \to a)) \leq \lambda(b \to a) = \lambda(b) \to \lambda(a)$. This means that $\varphi(a) \odot \lambda(b) = \lambda(a)$. $\square$

Theorem 4.10. *If φ is a λ-derivation on BL-algebra of L, then $\ker \varphi$ is closed under the operation " $\odot$ ".*

Proof. Suppose $a, b \in \ker \varphi$. Then $\varphi(a) = \varphi(b) = 0$. Thus, $\varphi(a \odot b) = \lambda(b) \odot (\varphi(a) \vee \varphi(b))$. Therefore, $\varphi(a \odot b) = (\lambda(b) \odot \varphi(a)) \vee (\lambda(b) \odot \varphi(b))$, hence $\varphi(a \odot b) \leq \varphi(a) \vee \varphi(b) = 0 \vee 0 = 0$. This means that $\varphi(a \odot b) = 0$, i.e., $a \odot b \in \ker \varphi$. Therefore, $\ker \varphi$ is closed with respect to " $\odot$ ". $\square$

5. Conclusion

In summary, this paper focuses on some notions in the field of BL-algebras. The primary objective was to introduce some special ideals within the framework of BL-algebras, unveiling their properties and establishing relationships between them. Additionally, we introduced a novel definition of $\odot$-derivation and λ-derivation for BL-algebras, and presented new and noteworthy properties associated with these derivations. We hope that the results of this paper can be more clarify the nature of ideals and $\odot$-derivation in BL-algebras and will be a foundation for further research and exploration in this algebraic structures.

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